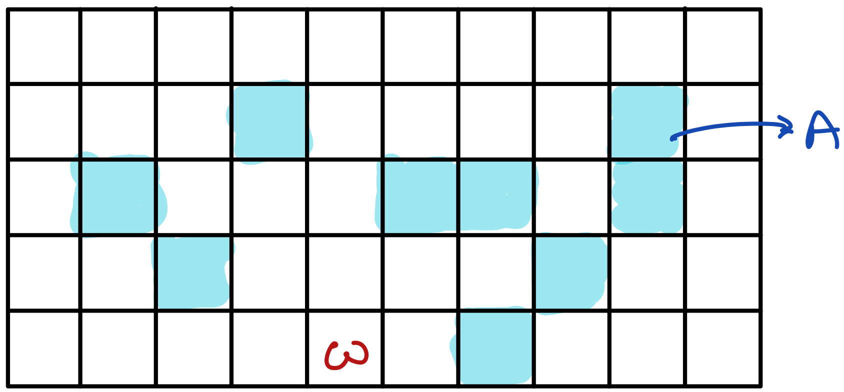
Recap



Probability Space : (Ω, ν)

$$\nu : \Omega \longrightarrow [0,1] \quad , \quad \sum_{\omega \in \Omega} \nu(\omega) = 1$$

Event : $A \subseteq \Omega$

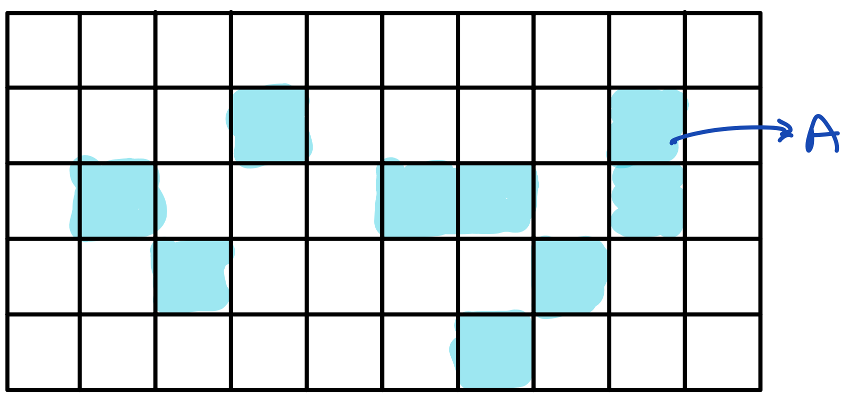
Random Variable : $X : \Omega \longrightarrow \mathbb{R}$

Expectation : $\mathbb{E}[X] = \sum_{\omega} \nu(\omega) \cdot X(\omega)$

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Conditioning

Conditioning on event $A \subseteq \Omega$ $\equiv$ Restricting probability space to A

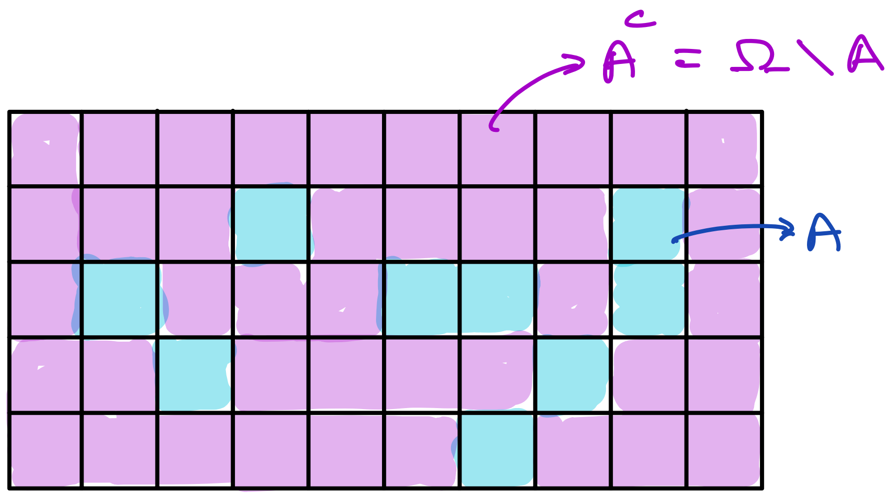


$$\nu_A(\omega) = \begin{cases} 0 & \text{if } \omega \notin A \\ \dfrac{\nu(\omega)}{\mathbb{P}[A]} & \text{if } \omega \in A \end{cases}$$

$$\mathbb{P}[B \mid A] = \sum_{\omega \in B \cap A} \nu_A(\omega) = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]}$$

$$\mathbb{E}[X \mid A] = \sum_{\omega} \nu_A(\omega)\, X(\omega) = \sum_{\omega} \frac{\nu(\omega)}{\mathbb{P}(A)} \cdot \mathbb{1}_A(\omega)\, X(\omega) = \frac{\mathbb{E}[\mathbb{1}_A(\omega) \cdot X(\omega)]}{\mathbb{P}[A]}$$

Total Probability and Total Expectation



- For any event $B \subseteq \Omega$

$$\mathbb{P}[B] = \mathbb{P}[A] \cdot \mathbb{P}[B \mid A] + \mathbb{P}[A^c] \cdot \mathbb{P}[B \mid A^c]$$

- For any (real-valued) r.v. X

$$\mathbb{E}[X] = \mathbb{P}[A] \cdot \mathbb{E}[X \mid A] + \mathbb{P}[A^c] \cdot \mathbb{E}[X \mid A^c]$$

- For any partition $A_1 \ldots A_k$ of Ω

$$\mathbb{E}[X] = \sum_{i=1}^{k} \mathbb{P}[A_i] \cdot \mathbb{E}[X \mid A_i]$$

Independence

Two events $A, B \subseteq \Omega$ are independent if

$$\mathbb{P}[A|B] = \mathbb{P}[A] \iff \mathbb{P}[A \wedge B] = \mathbb{P}[A] \cdot \mathbb{P}[B] \iff \mathbb{P}[B|A] = \mathbb{P}[B]$$

Two random variables X, Y on the same (finite) Ω are independent if $\forall x, y$ s.t. $\mathbb{P}[X = x] > 0$, $\mathbb{P}[Y = y] > 0$

the events $\{X = x\}$ and $\{Y = y\}$ are independent

- If $X, Y : \Omega \to \mathbb{R}$ are independent, then $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

<u>Proof</u>:

$$\mathbb{E}[X \cdot Y] = \sum_{\omega} \nu(\omega) \cdot X(\omega) Y(\omega) = \sum_{x,y} \Bigg(\sum_{\substack{\omega : X(\omega) = x \\ Y(\omega) = y}} \nu(\omega) \Bigg) x \cdot y$$

$$= \sum \mathbb{P}[X = x \wedge Y = y] \cdot x \cdot y$$

$$= \sum_{x,y} \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y]\, x \cdot y$$

Ex: Is the converse true?

Independence of multiple events

- $A_1, \dots, A_n \subseteq \Omega$ are mutually independent if

$$\forall S \subseteq [n], \quad \mathbb{P}\Big[\bigcap_{i \in S} A_i\Big] = \prod_{i \in S} \mathbb{P}[A_i]$$

only for $|S| \leq k \longrightarrow$ k-wise independence

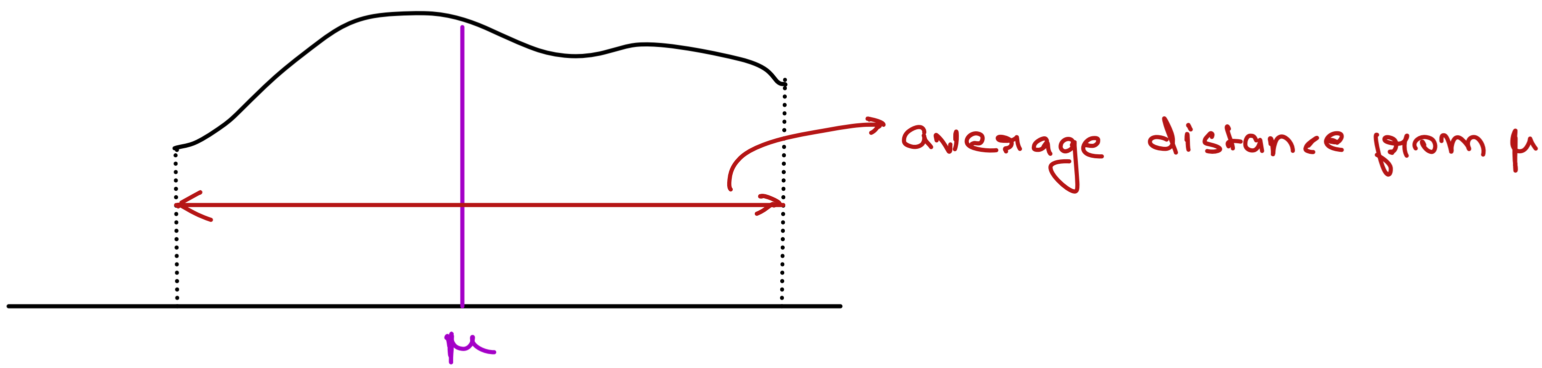
- $X_1, \dots, X_n : \Omega \longrightarrow \mathbb{R}$ are mutually independent if

$\forall x_1, \dots, x_n$ the events $\{X_1 = x_1\}, \dots, \{X_n = x_n\}$ are mutually independent.

Variance and covariance

- For a random variable $X : \Omega \to \mathbb{R}$ with $\mathbb{E}[X] = \mu$

$$Var[X] = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - (\mathbb{E}(X))^2$$



- For two random variables $X_1, X_2 : \Omega \to \mathbb{R}$, $\mathbb{E}[X_1] = \mu_1$ $\mathbb{E}[X_2] = \mu_2$

$$Cov[X_1, X_2] = \mathbb{E}[(X_1 - \mu_1)(X_2 - \mu_2)] = \mathbb{E}[X_1 X_2] - \mu_1 \mu_2$$

Ex: X_1, X_2 are independent $\Rightarrow Cov[X_1, X_2] = 0$

Dealing with infinite probability spaces (Ω)

Countable Ω : Same definitions

Care with convergence of $\sum_{\omega \in \Omega} \ldots$

Uncountable Ω : Needs more care

(after few lectures)

Bernoulli random variables

Bernoulli (p) $\quad X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$

$\Omega = \{0, 1\}$

$1-p$	p
0	1

$\mathbb{E}[X] = p$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \underbrace{\mathbb{E}X}_{\text{because } X^2 = X \text{ for } X \in \{0,1\}} - (\mathbb{E}[X])^2$$

$$= p - p^2 = p(1-p)$$

Binomial random variables

Binomial (n, p) $\qquad Z_n = \underbrace{X_1 + \cdots + X_n}$ $\rightarrow$ independent identical, Bernoulli (p) (i.i.d.)

$$\Omega = \{0,1\}^n$$

$$\mathbb{E}[Z_n] = \sum_i \mathbb{E}[X_i] = n \cdot p = \mu .$$

$$\mathrm{Var}[Z_n] = \mathbb{E}\left[\left(\sum X_i - \mu\right)^2\right]$$

$$= \mathbb{E}\left[\left(\sum_i (X_i - \mu_i)\right)^2\right] = \sum_i \underbrace{\mathbb{E}[(X_i - \mu_i)^2]}_{= p(1-p)} + \sum_{i \neq j} \underbrace{\mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]}_{= 0}$$

$$= n \cdot p \cdot (1-p)$$

Ex: $P[Z_n = k] = ?$

Geometric random variables

$\underbrace{Y}_{\text{Geometric}(p)}$ = position of first 1 among $\underbrace{X_1, X_2, X_3 \ldots\ldots}_{\text{i.i.d. Bernoulli}(p)}$

$\Omega = \{1, 01, 001, 0001, \ldots - - - - \}$

$A \equiv \{X_1 = 0\}$

$$E[Y] = \mathbb{P}(A) \cdot \mathbb{E}[Y \mid A] + \mathbb{P}[A^c] \cdot \mathbb{E}[Y \mid A^c]$$

$$= p \cdot 1 + (1-p) \cdot (\mathbb{E}[Y] + 1)$$

$$\Rightarrow \mathbb{E}[Y] = 1/p$$

Ex: $\text{Var}[Y] =$

Coupon collection

- n types of coupons
- get independent random type at each time step

$T \equiv$ first time we have all n types

$\mathbb{E}[T] =$

$$T = X_1 + X_2 + \cdots + X_n$$

$X_i \equiv$ time to go from $i-1$ types to i types

$$\equiv \text{Geometric}\left(\frac{n-(i-1)}{n}\right)$$

$$\mathbb{E}[X_i] = \frac{n}{n-i+1}$$

$$\mathbb{E}[T] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} = n\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \approx c \cdot n \cdot \ln n$$